TOPIC - 2nd order Differential Equation With variable co-efficients
(Method of changing the Independent variable)

B.Sc III

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Study Material →

Let $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ ——— (1)

be a differential Equation in which P, Q, R are functions of x.

Let $z = f(x)$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)$$

$$= \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \frac{dz}{dx}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2$$

Putting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1) le get

$$\left\{\frac{dy}{dz}\cdot\frac{d^2z}{dx^2}+\frac{d^2y}{dz^2}\cdot\left(\frac{dz}{dx}\right)^2\right\}+P\left[\frac{dy}{dz}\cdot\frac{dz}{dx}\right]+Qy=R$$

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2+\left\{P\frac{dz}{dx}+\frac{d^2z}{dx^2}\right\}\frac{dy}{dz}+Qy=R$$

$$\Rightarrow \frac{d^2y}{dz^2}+\frac{P\frac{dz}{dx}+\frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz}+\frac{Q}{\left(\frac{dz}{dx}\right)^2}\cdot y=\frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$\Rightarrow \frac{d^2y}{dz^2}+P_1\frac{dy}{dz}+Q_1y=R_1 \quad \text{——— (2)}$$

Where $P_1=\dfrac{P\frac{dz}{dx}+\frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2}$, $Q_1=\dfrac{Q}{\left(\frac{dz}{dx}\right)^2}$

$$R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}$$

there are two ways of choosing the relation $z=f(x)$

Case 1 - Let us choose z such that $P_1=0$

$$\therefore \frac{d^2z}{dx^2}+P\frac{dz}{dx}=0$$

put $\frac{dz}{dx} = q$

$$\therefore \frac{dq}{dx} + Pq = 0$$

$$\Rightarrow \frac{dq}{q} = -P\,dx$$

$$\Rightarrow \log q = -\int P\,dx$$

$$\Rightarrow q = e^{-\int P\,dx}$$

$$\Rightarrow \frac{dz}{dx} = e^{\int -P\,dx}$$

$$\therefore z = \int e^{\int -P\,dx}\,dx \qquad \text{③}$$

The above is the required relation between z and x which will make $P_1 = 0$ and the equation reduces to the form

$$\frac{d^2y}{dz^2} + Q_1 y = R_1$$

If Q_1 comes out to be constant with the help of relation ③ then the above equation reduces to linear equation with constant coefficients and is easily integrable.

Again if S_1 comes out to be of the form $\frac{K}{z^2}$ with the help of relation (3) then the above Equation reduces to the form of homogenous Equation -

Case II - Let us choose z such that Q_1 is constant, say $= a^2$

i.e $$\frac{S}{\left(\frac{dz}{dx}\right)^2} = a^2$$

$$\Rightarrow a\frac{dz}{dx} = \sqrt{S}$$

$$\Rightarrow dz = \frac{1}{a}\sqrt{S}\,dx$$

$$\Rightarrow z = \frac{1}{a}\int\sqrt{S}\,dx \quad \text{———} \quad (4.)$$

Above is the required relation between z and x which will make S_1 a constant and the equation reduces to the form

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + a^2y = R_1$$

If P_1 comes out to be a constant with the help of relation (4), then the above equation reduces to linear differential

Equation with constant co-efficients
it will be found that the
Equation can be solved by Either
way.